

## Theory of granular gases: some recent results and some open problems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2005 J. Phys.: Condens. Matter 17 S2591

(<http://iopscience.iop.org/0953-8984/17/24/015>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

### Download details:

IP Address: 129.252.86.83

The article was downloaded on 28/05/2010 at 05:01

Please note that [terms and conditions apply](#).

# Theory of granular gases: some recent results and some open problems

I Goldhirsch<sup>1,3</sup>, S H Noskowicz<sup>1</sup> and O Bar-Lev<sup>2</sup>

<sup>1</sup> Department of Fluid Mechanics and Heat Transfer, School of Mechanical Engineering, Faculty of Engineering, Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel

<sup>2</sup> School of Mathematics, Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel

E-mail: isaac@eng.tau.ac.il

Received 16 March 2005

Published 3 June 2005

Online at [stacks.iop.org/JPhysCM/17/S2591](http://stacks.iop.org/JPhysCM/17/S2591)

## Abstract

One of the goals of the present paper is to present a brief, and admittedly somewhat biased, review of some recent theoretical advances in the field of granular gases. Another goal is to highlight some challenges facing this field. A third goal is to present some new results concerning the Chapman–Enskog expansion. These include an extension to weakly frictional granular gases, and a study of a stationary granular gas subject to a heat flux (an earlier study describing a stationary sheared granular gas is presented as well). In addition, a computer-aided method for inverting the linearized Boltzmann operator, which should be useful for kinetic theorists, is presented in an appendix. A further goal is to briefly discuss possible extensions beyond Boltzmann kinetics, as well as beyond hydrodynamics, i.e., the moderately dense regime on one hand, and the Knudsen regime on the other hand. An attempt is made throughout this paper to put granular gases in a general context, by distinguishing between those features that are common to granular and molecular gases and those that are not. Also, stress is put on the physics (and methods) rather than phenomena. While this is not a review article, parts of it are intended for the uninitiated in the field.

## 1. Introduction

Granular gases [1] are collections of macroscopic solid particles that interact by inelastic collisions. The term ‘macroscopic’ is taken to imply that

- (i) the size of a grain is sufficiently large (say, larger than about a micron) to render Brownian motion practically inconsequential (when the grains are embedded in an ambient fluid) and

<sup>3</sup> Author to whom any correspondence should be addressed.

- (ii) most or all of the energy that is considered to be ‘lost’ into the internal degrees of freedom of a grain during a collision is not retrievable in a practically finite time due to the large number of these degrees of freedom.

The term ‘collision’ implies that the grain interactions are short ranged, and that the contact time during a collision (the collision time) is short with respect to e.g., the mean free time. In addition, it is often assumed that all collisions are binary (see however [2]). This is actually true for the idealized case of hard sphere collisions, but only an approximation in reality, that becomes better the more dilute the gas is. Furthermore, when the ambient fluid in which the grains may be embedded is of consequence, the system comprises a suspension; following Bagnold (see e.g., review [3]), when the ratio of the stress due to the grains and that due to the fluid is sufficiently large (say, above 400), the effects of the fluid can be ignored.

On Earth granular matter needs to be fluidized by external forcing (such as vertical vibration, horizontal shaking or shearing), else it is in a solid state. Gravity can cause fluidization in rock slides and snow avalanches. In all of the above cases the granular gases are in nonequilibrium states. This is the first and simplest consequence of the dissipative nature of the grain interactions. Another, perhaps less obvious, consequence is the lack of scale separation in granular gases. This property, as well as other consequences of inelasticity, is taken up in the next section.

## 2. Some consequences of inelasticity and dimensions

The concept of a granular gas is superficially similar to that of the classical model of a molecular gas. There are however important differences between these two ‘kinds’ of gases. The first difference is of practical nature: due to the fact that the grains are of macroscopic dimensions, their typical number in an actual granular system is far smaller than the Avogadro number. Therefore fluctuations in granular gases are expected to be more prominent than in molecular gases. However, if size were the only difference a (theoretically) sufficiently large sized granular gas would have exhibited properties similar to those of a molecular gas, albeit on larger scales. The inelasticity of the grain interactions dictates much deeper differences, some of which are the following.

- (i) As mentioned in the introduction, all granular gases which have finite kinetic energy must be in non-equilibrium states, since energy must be pumped into the granular gas to compensate for the losses incurred in the inelastic collisions.
- (ii) Statistical fluctuations, instabilities or external forces may beget density inhomogeneities in a granular gas. Since in relatively dense domains the rate of collisions (proportional to the square of the number density) is higher than in dilute domains, the kinetic energy in the dense domains decays at a higher rate than in the dilute ones. The ensuing pressure difference leads grains from the dilute into the dense domains, thereby further increasing the density of the latter, and giving rise to dense clusters [4–7]. It follows that no granular gas can be found in a truly homogeneous state (such as molecular equilibrium, in the absence of gravity). In unforced granular systems the clusters may merge in a process of coarsening [8]. In sheared granular gases [7] an instability creates a density nonuniformity that begets clusters. The latter collide with each other and disperse, and are then recreated by the same mechanism. Sufficiently small granular systems will not exhibit clustering [5], but even then there are linear instabilities which render the system inhomogeneous [4–6, 9].
- (iii) Most states of granular matter in general, and granular gases in particular, are metastable and history dependent. This property can be related in part to clustering. For instance,

upon increasing the shear rate in a uniform granular Couette system, one injects energy at the boundaries, thus raising the granular temperature (defined as the average of the square of the fluctuating velocity), hence the pressure, there. As a result, material moves away from the boundaries toward the ‘centre’ of the system, where clusters and/or a central plug are formed. More states of granular Couette systems can be created by varying the initial conditions [7].

- (iv) A ball hitting a floor with vertical velocity  $v$  is known to bounce off with a velocity  $ev$ , where  $e$  is the coefficient of restitution. When the ball is dropped from rest at height  $h_0$ , its  $n$ th maximal height is  $e^{2n}h_0$ . The time that elapses between positions  $h_n$  and  $h_{n+1}$  of the ball is given by  $\tau_n = \tau_0 e^n$ . Since the sum of  $\tau_n$  is finite (as  $0 < e < 1$ ), it follows that an infinite number of collisions can occur in a finite time. A similar process, known as ‘inelastic collapse’, may take place in granular gases [10, 11], leading (via a theoretically infinite number of collisions) to the emergence of strings of particles whose relative velocities vanish. For a review, see [12]. Clearly, ‘collapse’ is a non-hydrodynamic phenomenon. In reality,  $e \rightarrow 1$  when the relative velocity of the colliding particles is sufficiently low; therefore, the collapse process is never completed.
- (v) Scale separation, between the microscopic (grain) scales and the macroscopic scales, in granular gases is weak or nonexistent [13]. This is demonstrated here using the example of a simple shear flow of a monodisperse collection of spheres, with a fixed coefficient of normal restitution,  $e$ . The velocity field is given by  $\mathbf{V} = \gamma y \hat{\mathbf{x}}$ , where  $\gamma$  is the shear rate,  $x$  is the streamwise coordinate, and  $y$  a spanwise coordinate. In the absence of gravity,  $\gamma^{-1}$  provides the only ‘input’ variable that has dimensions of time. Recalling that the granular temperature has dimensions of squared velocity, it follows from dimensional analysis (also from kinetic calculations, see below) that  $T \propto \gamma^2 \ell^2$ , where  $\ell$  is the mean free path (the only relevant microscopic length scale). We define the *degree of inelasticity*,  $\epsilon$ , by  $\epsilon \equiv 1 - e^2$ . Clearly,  $T$  should be larger, for a given value of  $\gamma$ , the smaller  $\epsilon$  is. Furthermore, when  $\epsilon = 0$ , the shear work raises  $T$  indefinitely (in molecular gases the excess energy escapes in the form of a heat flux through the boundaries; there is no full equivalent to this in granular systems). Therefore  $T \rightarrow \infty$  as  $\epsilon \rightarrow 0$ . While this argument does not provide the precise form of this divergence, an initial guess (corroborated by mean field, as well as detailed kinetic calculations [14–16]) would be that  $T \propto \frac{1}{\epsilon}$ . All in all, one obtains  $T = C \frac{\gamma^2 \ell^2}{\epsilon}$ . The value of  $C$  is about 1 in two dimensions and 3 in three dimensions. It follows that  $\frac{\gamma \ell}{\sqrt{T}} = \frac{\sqrt{\epsilon}}{\sqrt{C}}$ , i.e. the change of speed over a mean free path (in the spanwise direction) is comparable to the thermal speed (unless  $\epsilon$  is very small), hence the shear rate can be considered ‘large’, and the system is typically supersonic. Indeed, shocks are a frequent occurrence in granular gases, see [1] for references. Consider next the mean free time,  $\tau$ , i.e. the ratio of the mean free path and the thermal speed:  $\tau \equiv \frac{\ell}{\sqrt{T}}$ . Clearly, in the simple shear flow  $\tau$  and  $\gamma^{-1}$  are the microscopic and macroscopic timescales characterizing the system, respectively. Since  $\frac{\tau}{\gamma^{-1}} = \tau \gamma = \frac{\ell \gamma}{\sqrt{T}} = \frac{\sqrt{\epsilon}}{\sqrt{C}}$ , there is no good temporal scale separation except in near-elastic cases. Consequently, one cannot *a priori* employ the assumption of ‘fast local equilibration’ and/or use local equilibrium as a zeroth order distribution function (both for solving the Boltzmann equation and for the study of generalized hydrodynamics of these systems) unless the system is nearly elastic.
- (vi) It follows from (v) that in the Chapman–Enskog (CE) expansion method (in ‘powers of the gradients’) one cannot neglect higher order gradient contributions (e.g., Burnett [16]) to granular hydrodynamics, except when  $\epsilon \ll 1$ . The Burnett equations are ill posed (but one can use them for steady states, else a resummation may be needed [17]). At finite

densities the Burnett coefficients diverge [18], possibly implying that the correct theory is non-analytic in the gradients [19], hence nonlocal.

- (vii) A molecular fluid that is not subject to strong thermal or velocity gradients possesses a range, or *plateau*, of scales (larger than the mean free path and far smaller than the scales characterizing macroscopic gradients) which can be used to define ‘scale independent’ field densities (e.g., mass density, stress). Such *plateaus* do not exist in systems which lack scale separation. Consequently, some of the fields (e.g., the stress tensor) characterizing granular gases may be scale (or resolution) dependent [20].
- (viii) In the example of the simple shear flow, consider the ratio of the  $xx$  component of the stress,  $\tau_{xx}$ , and the pressure,  $p \equiv \frac{\tau_{xx} + \tau_{yy} + \tau_{zz}}{3}$ . As this is a ( $\gamma$  dependent) dimensionless entity, which must be even in  $\gamma$  by symmetry, it follows that to second order in  $\gamma$ :  $\frac{\tau_{xx}}{p} = \frac{1}{3}(1 + c_{xx} \frac{\gamma^2 \ell^2}{T})$ , in three dimensions, where  $c_{xx}$  is dimensionless. Notice that the  $\mathcal{O}(\gamma^2)$  correction is a Burnett contribution (second order in the gradients). A similar result holds for  $\tau_{yy}$ . The two constants,  $c_{xx}$  and  $c_{yy}$ , are not required to be equal; indeed, detailed kinetic calculations show that they are both  $\mathcal{O}(1)$  and different from each other. This results in significant normal stress differences. Notice that in molecular gases the typical value of  $\frac{\gamma^2 \ell^2}{T}$  is negligibly small (for air at STP conditions and  $\gamma = 0.1 \text{ s}^{-1}$ , the value is about  $10^{-18}$ ). This shows yet another facet of granular gases: they may ‘amplify’ some negligible effects (in molecular gases) to the level of  $\mathcal{O}(1)$  phenomena.
- (ix) As granular gases at finite granular temperature are in non-equilibrium states, one should not expect them to possess Maxwellian velocity distributions or exhibit equipartition. On the contrary, one should probably be surprised when the distribution is (close to) Maxwellian or equipartition is approximately obeyed.

As mentioned, one of the practical implications of the macroscopic grain sizes is the enhancement of fluctuations in granular gases. There is also an experimental advantage to the size, namely the fact that (at least part of) the dynamics of a granular gas is visible to the naked eye; therefore, one can, e.g., study the inner structure of a shock by ‘just’ using a camera [21] (in air the typical shock thickness is of the order of a tenth of a micron).

### 3. Kinetic theory of granular gases

#### 3.1. Time dependent formulation

The derivation of the classical Boltzmann equation can be modified to account for inelastic collisions [16, 22]. A basic assumption in the derivation, i.e., that the density is low, can be easily theoretically conceived and experimentally realized. However, the assumption of molecular chaos, even for low densities, is not as justified for granular gases as for molecular gases. The reason is that the normal component of the relative velocity is reduced by a factor  $e$  upon collision, thereby rendering the particle velocities more correlated. Indeed, such correlations have been noted in molecular dynamic (MD) simulations [1, 4–7, 23]. As the coefficient of restitution approaches unity, these correlations become smaller. This implies that the Boltzmann equation for granular gases should apply (at best) to near-elastic collisions. Since the CE expansion is valid only when there is scale separation, it follows that both the kinetics and hydrodynamics of granular gases are restricted, at least in principle, to the case of near-elastic collisions.

Consider a collection of monodisperse hard spheres, of unit mass, and diameter  $d$ , whose collisions are characterized by a constant coefficient of normal restitution,  $e$ . The binary

collision between spheres  $i$  and  $j$  results in the following velocity transformation:

$$\mathbf{v}_i = \mathbf{v}'_i - \frac{1+e}{2} (\hat{\mathbf{k}} \cdot \mathbf{v}'_{ij}) \hat{\mathbf{k}} \quad (1)$$

where  $(\mathbf{v}'_i, \mathbf{v}'_j)$  are the precollisional velocities,  $(\mathbf{v}_i, \mathbf{v}_j)$  are the corresponding postcollisional velocities,  $\mathbf{v}'_{ij} \equiv \mathbf{v}'_i - \mathbf{v}'_j$ , and  $\hat{\mathbf{k}}$  is a unit vector pointing from the centre of sphere  $i$  to that of sphere  $j$  at the moment of contact. Note that the normal component of the relative velocity of two colliding particles is reduced upon collision by a factor  $e$ .

Let  $f(\mathbf{v}, \mathbf{r}, t)$  denote the single-particle (or singlet) distribution function, i.e. the number density of particles having velocity  $\mathbf{v}$  at a point  $\mathbf{r}$ , at time  $t$ . The Boltzmann equation, corresponding to the above model, reads [16, 22]

$$\frac{\partial f(\mathbf{v}_1)}{\partial t} + \mathbf{v}_1 \cdot \nabla f(\mathbf{v}_1) = d^2 \int_{\hat{\mathbf{k}} \cdot \mathbf{v}_{12} > 0} d\mathbf{v}_2 d\hat{\mathbf{k}} (\hat{\mathbf{k}} \cdot \mathbf{v}_{12}) \left( \frac{1}{e^2} f(\mathbf{v}'_1) f(\mathbf{v}'_2) - f(\mathbf{v}_1) f(\mathbf{v}_2) \right), \quad (2)$$

where  $\nabla$  is a gradient with respect to  $\mathbf{r}$ . The unit vector  $\hat{\mathbf{k}}$  points from the centre of particle '1' to the centre of particle '2'. The dependence of  $f$  on  $\mathbf{r}$  and  $t$  is not explicitly spelled out in equation (2), for the sake of notational simplicity. Notice that in addition to the explicit dependence of equation (2) on  $e$ , it also implicitly depends on  $e$  through the relation between the postcollisional and precollisional velocities. The condition  $\hat{\mathbf{k}} \cdot \mathbf{v}_{12} > 0$  represents the fact that only particles that approach each other can collide.

In many-body systems which possess scale separation, one identifies the densities of the conserved entities as hydrodynamic fields, whose dynamics is slow. When these fields barely change on the spatial scale of a mean free path ('small' gradients) or the temporal scale of a mean free time, the system can 'locally equilibrate' while the slow fields are practically fixed. This renders local equilibrium a good 'zeroth order' distribution. The latter has the same dependence on the values of the local fields as equilibrium has on the global fields. Note that one can exchange the set of hydrodynamic fields for an equivalent 'non-slow' set, which depends on the original fields (such as the temperature field). In monodisperse systems the set of slow or hydrodynamic fields consists of the number density,  $n(\mathbf{r}, t)$  (or mass density,  $\rho(\mathbf{r}, t)$ ), the momentum density  $\mathbf{p}(\mathbf{r}, t)$  (hence, one can use the velocity field,  $\mathbf{V}(\mathbf{r}, t) \equiv \mathbf{p}(\mathbf{r}, t)/\rho(\mathbf{r}, t)$  instead), and the energy density (for hard objects, this is equivalent to the kinetic energy density; by subtracting the macroscopic kinetic energy density,  $\frac{1}{2}\rho V^2$  from the 'full' energy density one obtains the fluctuating energy density). In the realm of granular gases,  $n$  and  $\mathbf{p}$  are 'slow', but the energy density is not strictly conserved. However, when the degree of inelasticity is sufficiently small, it is justified to include it (or the granular temperature,  $T(\mathbf{r}, t)$ ) as a hydrodynamic field, as the energy density is nearly a slow field. Furthermore, it comprises an important characterization of the state of a granular gas. An additional argument is presented below. The aforementioned fields are moments of the single-particle distribution:  $n(\mathbf{r}, t) \equiv \int d\mathbf{v} f(\mathbf{v}, \mathbf{r}, t)$ ,  $\mathbf{V}(\mathbf{r}, t) \equiv \frac{1}{n(\mathbf{r}, t)} \int d\mathbf{v} \mathbf{v} f(\mathbf{v}, \mathbf{r}, t)$ , and  $T(\mathbf{r}, t) \equiv \frac{1}{n(\mathbf{r}, t)} \int d\mathbf{v} (\mathbf{v} - \mathbf{V})^2 f(\mathbf{v}, \mathbf{r}, t)$ , respectively. It is *a priori* unclear whether these fields are sufficient for a proper closure of the hydrodynamic equations of motion for granular gases, since one cannot naively extrapolate from the case of molecular gases, but this turns out to be the case (within the framework of the CE expansion). The equations of motion for the hydrodynamic fields can be derived by multiplying the Boltzmann equation, equation (2), by 1,  $\mathbf{v}_1$  and  $v_1^2$  respectively, and integrating over  $\mathbf{v}_1$ . A standard procedure [24–26] (which employs the symmetry properties of the collision integral on the right-hand side of the Boltzmann equation) yields [15]

$$\frac{Dn}{Dt} + n \frac{\partial V_i}{\partial r_i} = 0, \quad (3)$$

$$n \frac{DV_i}{Dt} + \frac{\partial P_{ij}}{\partial r_j} = 0, \quad (4)$$

$$n \frac{DT}{Dt} + 2 \frac{\partial V_i}{\partial r_j} P_{ij} + 2 \frac{\partial Q_j}{\partial r_j} = -n\Gamma. \quad (5)$$

Denote by  $\mathbf{u} \equiv \mathbf{v} - \mathbf{V}$  the fluctuating (or peculiar) velocity. The fluxes in equations (3)–(5) are given as follows:  $P_{ij} \equiv n \langle u_i u_j \rangle$  is the stress tensor, and  $Q_j \equiv \frac{1}{2} n \langle u^2 u_j \rangle$  is the heat flux vector, where  $\langle \rangle$  denotes an average with respect to  $f$ . In addition,  $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla$  is the material derivative, and  $\Gamma$ , which accounts for the energy loss in the (inelastic) collisions, is given by  $\Gamma \equiv \frac{\pi(1-e^2)d^2}{8n} \int d\mathbf{v}_1 d\mathbf{v}_2 v_{12}^3 f(\mathbf{v}_1) f(\mathbf{v}_2)$ . Equations (3)–(5) are *exact* consequences of the Boltzmann equation. They also comprise the equations of continuum mechanics, and thus their validity is more general than that of the Boltzmann equation [20]. The above ( $\langle \rangle$ ) averages, which determine the constitutive relations, follow from the Boltzmann equation and are therefore limited to low density (for corrections see, e.g., [20]).

Local equilibrium is a solution of the Boltzmann equation for  $\epsilon = 0$ , when gradients are neglected. As no such solution exists for granular gases, the CE method needs to be modified in this case. There are at present two (systematic) generalized CE expansions for granular gases. The method proposed in [27] is based on an expansion in the Knudsen number (gradients) around a local homogeneous cooling state (HCS, see e.g., [28])<sup>4</sup>. Formally, this is a solution of the inelastic Boltzmann equation at zero Knudsen number. The method does not formally restrict the value of  $\epsilon$  to be small, hence, in principle, it is correct for all values of this parameter. However, as explained above, this cannot be the case because of the lack of scale separation. This fact notwithstanding, the constitutive relations obtained this way are claimed [27] to agree with DSMC simulations for values of  $e$  as low as 0.6. They also agree, for small  $\epsilon$ , with the results presented below.

The classical Chapman–Enskog expansion assumes the smallness of the Knudsen number,  $K \equiv \frac{\ell}{L}$ , where  $\ell$  is the mean free path given by  $\ell = \frac{1}{\pi n d^2}$ , and  $L$  is a macroscopic length scale i.e., the length scale which is resolved by hydrodynamics, not necessarily the system size. Here we employ  $\epsilon$  as a second small parameter. Note that the double limit  $\epsilon \rightarrow 0$  and  $K \rightarrow 0$ , with constant number density, corresponds to a homogeneous, elastically colliding collection of spheres for which the distribution function is Maxwellian. This limit is not singular (local equilibration occurs on a timescale of a few mean free times; during such time a small degree of inelasticity has a negligible effect). Consequently, the local equilibrium distribution function can be used as zeroth order in a perturbation theory in  $K$  and  $\epsilon$ . It is convenient to perform a rescaling of the Boltzmann equation, as follows: spatial gradients are rescaled as  $\nabla \equiv \frac{1}{L} \tilde{\nabla}$ , the rescaled peculiar velocity (in terms of the thermal speed) is  $\tilde{\mathbf{u}} \equiv \sqrt{\frac{3}{2T}} (\mathbf{v} - \mathbf{V})$ , and  $f \equiv n \left(\frac{3}{2T}\right)^{\frac{3}{2}} \tilde{f}(\tilde{\mathbf{u}})$ . In terms of the rescaled quantities, the Boltzmann equation assumes the form

$$\begin{aligned} & \tilde{\mathbf{D}} \tilde{f} + \tilde{f} \tilde{\mathbf{D}} (\log n - \frac{3}{2} \log T) \\ &= \frac{1}{\pi} \int_{\hat{\mathbf{k}} \cdot \tilde{\mathbf{u}}_{12} > 0} d\tilde{\mathbf{u}}_2 d\hat{\mathbf{k}} (\hat{\mathbf{k}} \cdot \tilde{\mathbf{u}}_{12}) \left( \frac{1}{e^2} \tilde{f}'(\tilde{\mathbf{u}}'_1) \tilde{f}'(\tilde{\mathbf{u}}'_2) - \tilde{f}(\tilde{\mathbf{u}}_1) \tilde{f}(\tilde{\mathbf{u}}_2) \right) \equiv \tilde{\mathbf{B}}(\tilde{f}, \tilde{f}, e), \end{aligned} \quad (6)$$

<sup>4</sup> The HCS is a scaling solution,  $f_{\text{HCS}}$ , of the inelastic Boltzmann equation corresponding to an unforced system, and a homogeneous (number) density, in which the time dependence of the distribution is determined by that of the granular temperature:  $f_{\text{HCS}}(t) = F(T(t))$ . The local HCS solution is one where the peculiar velocity distribution at every point (and time) is the same as in the HCS solution, but the density and granular temperature can be space dependent, corresponding to the local values of the hydrodynamic fields (as in local equilibrium).

where  $\tilde{\mathbf{D}} \equiv K\sqrt{\frac{3}{2T}}(L\frac{\partial}{\partial t} + \mathbf{v} \cdot \tilde{\nabla})$ . Notice that  $\tilde{\mathbf{D}}$  is not a material derivative since  $\mathbf{v}$  is not the hydrodynamic velocity but rather the particle's velocity. Next, one can express the (rescaled) distribution function,  $\tilde{f}$ , as follows:  $\tilde{f}(\tilde{\mathbf{u}}) = \tilde{f}_0(\tilde{u})(1 + \Phi)$ , where  $\tilde{f}_0(\tilde{u}) = \frac{1}{\pi^{\frac{3}{2}}}e^{-\tilde{u}^2}$ . Assume that  $\Phi$  can be expanded in both small parameters,  $\epsilon$  and  $K$ , as follows:  $\Phi = \Phi_K + \Phi_\epsilon + \Phi_{KK} + \Phi_{K\epsilon} + \dots$ , where subscripts indicate the order of the corresponding terms in the small parameters, e.g.,  $\Phi_K = \mathcal{O}(K)$ . In parallel, the operation of  $\tilde{\mathbf{D}}$  on any function of the field variables,  $\psi$ , can be formally expanded as the following sum:  $\tilde{\mathbf{D}}\psi = \tilde{\mathbf{D}}_K\psi + \tilde{\mathbf{D}}_\epsilon\psi + \tilde{\mathbf{D}}_{KK}\psi + \tilde{\mathbf{D}}_{K\epsilon}\psi + \tilde{\mathbf{D}}_{\epsilon\epsilon}\psi + \dots$ , where, e.g.,  $\tilde{\mathbf{D}}_{K\epsilon}\psi$  is the  $\mathcal{O}(K\epsilon)$  term in the expansion of  $\tilde{\mathbf{D}}\psi$  in powers of  $K$  and  $\epsilon$ . Since this expansion is well defined, we shall refer to the symbols  $\tilde{\mathbf{D}}_K$ ,  $\tilde{\mathbf{D}}_\epsilon$  etc as operators in their own right.

Upon substituting  $e = 1$  (or  $\epsilon = 0$ ) in the right-hand side of the Boltzmann equation, and retaining only  $\mathcal{O}(K)$  terms, one obtains, after some algebra [16],

$$\tilde{\mathbf{L}}(\Phi_K) = 2K\overline{\tilde{u}_i\tilde{u}_j}\sqrt{\frac{3}{2T}}\frac{\partial V_i}{\partial \tilde{r}_j} + K\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial \log T}{\partial \tilde{r}_i}. \quad (7)$$

An overline denotes a symmetrized traceless tensor, i.e.,  $\overline{A_{ij}} \equiv \frac{1}{2}(A_{ij} + A_{ji}) - \frac{1}{3}A_{kk}\delta_{ij}$ . The linear inhomogeneous equation (7), which is identical to the  $\mathcal{O}(K)$  equation for an elastic hard sphere gas (since  $\epsilon$  does not affect this order), is soluble only when the Fredholm alternative (or solubility condition) is satisfied, i.e. its right-hand side is orthogonal to the (left) eigenfunctions of the operator  $\tilde{\mathbf{L}}$ , which have vanishing eigenvalues:  $1$ ,  $\mathbf{u}$  and  $u^2$ , i.e. the conserved entities at a collision. The isotropy and linearity of the operator  $\tilde{\mathbf{L}}$  [24–26] imply that the solution of equation (7) is of the form

$$\Phi_K(\tilde{\mathbf{u}}) = 2K\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\sqrt{\frac{3}{2T}}\frac{\partial V_i}{\partial \tilde{r}_j} + K\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial \log T}{\partial \tilde{r}_i}, \quad (8)$$

where  $\tilde{u} \equiv \|\tilde{\mathbf{u}}\|$ . It is common to expand  $\hat{\Phi}_v$ ,  $\hat{\Phi}_c$ , and similar functions in (truncated) series of Sonine polynomials (for another method, see [16]). An accurate solution of this equation (as well as the other equations arising in the CE expansion) is hard to obtain, since the integrals involving high order Sonine polynomials are nontrivial. A computationally aided method, which alleviates this difficulty, is presented in appendix A.

Since, by construction, the local equilibrium distribution function,  $f_0$ , corresponds to the actual values of the macroscopic fields, the perturbative corrections must not change the values of these fields. This implies that (to each order in perturbation theory)  $\Phi$  *should be orthogonal (with respect to the weight function  $f_0$ ) to the moments that define these fields, which are also the collisional invariants, as well as the invariants of the (linearized) Boltzmann operator:  $1$ ,  $\tilde{\mathbf{u}}$  and  $\tilde{u}^2$* . The above condition is the reason the (generalized) CE expansion can be systematically carried out to all orders in the small parameters [16], as the same conditions are also the solubility conditions of the linearized Boltzmann equation, at each order in the expansion.

The above method yields the following constitutive relations, presented below to first order in  $K$  and  $\epsilon$  (see [16] for the Burnett contributions). The heat flux assumes the form  $Q_i = -\tilde{\kappa}n\ell\sqrt{T}\frac{\partial T}{\partial r_i} - \tilde{\lambda}\ell\sqrt{T^3}\frac{\partial n}{\partial r_i} + \text{Burnett terms}$ , where  $\tilde{\kappa} \approx 0.4101 + 0.1072\epsilon + \mathcal{O}(\epsilon^2)$  and  $\tilde{\lambda} \approx 0.2110\epsilon + \mathcal{O}(\epsilon^2)$ . Notice that the heat flux includes a ‘non-Fourier’ term at order  $K\epsilon$ , which is proportional to the density gradient and to  $\epsilon$ . This term had been first discovered in [16], rediscovered shortly thereafter in [27], and further rediscovered on the basis of numerical simulations in [29]. It is of importance in applications, e.g., in the determination of the temperature and density profile of a vertically vibrated granular system [30]. The stress tensor reads  $P_{ij} = \frac{1}{3}nT\delta_{ij} - 2\tilde{\mu}n\ell\sqrt{T}\frac{\partial V_i}{\partial r_j} + \text{Burnett terms}$ , where

$\tilde{\mu} \approx 0.3249 + 0.0576\epsilon + \mathcal{O}(\epsilon^2)$ . The inelastic dissipation term,  $\Gamma$ , reads  $\Gamma = \frac{\tilde{\delta}}{\ell} T^{\frac{3}{2}} +$  Burnett terms, where [16]  $\tilde{\delta} \approx \sqrt{\frac{16}{27\pi}}\epsilon - 0.0112\epsilon^2$ . The normal stress difference (between  $P_{xx}$  and  $P_{yy}$ , normalized by their average), calculated from the Burnett terms (not shown here) equals 0.45 for  $e = 0.8$  and 0.88 for  $e = 0.6$ , in good agreement with numerical results [31]: 0.42 and 0.86, respectively (for a volume fraction of  $\nu = 0.025$ ).

One of the conclusions from the above method concerns the choice of the slow fields for use in the CE expansion. These fields are not necessarily the exactly conserved fields, but rather those fields that are conserved in the limit which defines the zeroth order solution; in our case, when  $\epsilon \rightarrow 0$ , the standard hydrodynamic fields become conserved.

The expansion in powers of  $\epsilon$  is at best asymptotic. If it were convergent, there would have been a radius of convergence within which one could have obtained a convergent theory for negative values of  $\epsilon$ . However, the latter corresponds to granular gases which ‘produce energy’ upon collision, hence their corresponding fields would diverge (with time).

### 3.2. Frictional granular gases

Most models of granular gases ignore the frictional interactions of the grains, even though all grains experience friction. The reason could be that it is not trivial to model frictional restitution; worse, it is hard to develop hydrodynamic equations of motion which account for friction (see however [32–35]). In addition, it is noteworthy that frictionless models have encountered numerous successes in describing some aspects of the dynamics of granular gases [1]. This may indicate that tangential restitution may not be very important for the description of some properties of granular gases. However, it is known that some aspects of granular gas dynamics are strongly dependent on friction, cf, e.g., the recent study of the effects of friction on granular patterns [36], and friction induced hysteresis [37]). Furthermore, it is known that friction enhances non-equipartition in the homogeneous cooling state; cf, e.g., [38–42]. It is therefore curious that a rather small proportion of the granular literature is devoted to frictional granular hydrodynamics; cf, e.g., [32, 43–45].

The study of gases whose constituents experience frictional interactions started (in 1894!) in the realm of molecular gases [25, 46]. Applications include granular celestial systems [47, 48]. In previous kinetic theoretical based studies of granular hydrodynamics [32, 44, 45] it is assumed that the basic distribution function is Maxwellian in both the velocity and angular velocity, and corrections due to gradients are identified (on the basis of symmetry). The assumed distribution function is substituted in the Enskog equations<sup>5</sup> [49], resulting in a closure for the constitutive relations. The above distribution corresponds to the rough elastic limit, which is appropriate for e.g., rough molecules. Since the basic model used for the description of granular gases is that of *smooth* particles (usually spheres), our goal is to study the case of weak friction, as a perturbation around the smooth limit. Specifically, we consider, as in the above, a monodisperse system of spherical grains of mass  $m = 1$ , diameter  $d$ , and moment of inertia  $I$  (for homogeneous spheres,  $I = \frac{2}{5}(\frac{d}{2})^2$ ), each. Denote the radius of gyration of a grain by  $\kappa$  (with  $\kappa \equiv \frac{4I}{d^2}$ ). The velocity of particle ‘ $i$ ’ is denoted by  $\mathbf{v}_i$ , and its angular momentum by  $\boldsymbol{\omega}_i$ . It is convenient to define a ‘spin variable’  $\mathbf{s}_i \equiv \frac{d}{2}\boldsymbol{\omega}_i$ , which has dimensions of speed.

Consider a binary collision between sphere ‘1’ and sphere ‘2’. Let  $\mathbf{k}$  be a unit vector pointing from the centre of sphere ‘2’ to the centre of sphere ‘1’. The relative velocity of

<sup>5</sup> The Enskog equations are equations for averages of single-particle properties, which can be derived from the Boltzmann equation, or by direct considerations [49]. They should not be confused with the Enskog–Boltzmann equation, which is a modified Boltzmann equation, designed to account for finite density effects.

sphere ‘1’ with respect to sphere ‘2’, at the point of contact (when they are in contact), is  $\mathbf{g}_{12} = \mathbf{v}_{12} + \mathbf{k} \times \mathbf{s}_{12}$ , where  $\mathbf{v}_{12} \equiv \mathbf{v}_1 - \mathbf{v}_2$ , and  $\mathbf{s}_{12} \equiv \mathbf{s}_1 + \mathbf{s}_2$ . In the following, precollisional entities are primed. During a collision the normal component of the relative velocity changes according to  $\mathbf{k} \cdot \mathbf{g}_{12} = -e \mathbf{k} \cdot \mathbf{g}'_{12}$ , where  $e$  is the coefficient of normal restitution. The effect of the tangential impulse at a collision is modelled as in [40, 45], with slight notational differences. The underlying physics of this model is as follows. Let  $\gamma$  be the angle between  $-\mathbf{k}$  and the relative velocity of the grains at the point of contact:  $\cos \gamma \equiv -\frac{\mathbf{k} \cdot \mathbf{g}'_{12}}{g'_{12}} = -\frac{\mathbf{k} \cdot \mathbf{v}'_{12}}{v'_{12}}$ , where  $g'_{12} \equiv \|\mathbf{g}'_{12}\|$ ; clearly,  $0 \leq \gamma \leq \frac{\pi}{2}$ . When  $\gamma$  is small, the direction of the relative velocity of the colliding particles, at the point of contact, is close to that in a head-on collision. One can imagine that the particles temporarily stick to each other (as in a rough particle collision), then are released with a relative tangential velocity which is a fraction (positive or negative) of the original relative tangential velocity. This fraction, denoted by  $\beta_0$  (with  $-1 \leq \beta_0 \leq 1$ , else energy can be ‘created’ in the collisions), is assumed to be constant in the model (though it really is not). The physical picture behind the possible reversal of the relative tangential velocity ( $\beta \approx 1$ ) will not be explained here (see e.g., [50] and references therein). In the case when  $\gamma$  is large, one can model the frictional interaction by assuming that the tangential part of the impulse is the product of the (kinetic) friction coefficient,  $\mu$ , and the normal (i.e. the direction  $\mathbf{k}$ ) part of the impulse. Define  $\gamma_0$  such that if  $\gamma > \gamma_0$  there is sliding (Coulomb friction) during the collision, whereas the above described ‘sticking’ occurs for  $\gamma < \gamma_0$ . Straightforward algebra shows that in both cases one can write (for the tangential part of the relative velocity at contact)  $\mathbf{k} \times (\mathbf{k} \times \mathbf{g}_{12}) = -\beta(\gamma) \mathbf{k} \times (\mathbf{k} \times \mathbf{g}'_{12})$ , where  $\beta(\gamma) = \min\{\beta_0, -1 + \frac{1+\kappa}{\kappa}(1+e)\mu \cot \gamma\}$ . The latter formula is obtained by demanding continuity of  $\beta(\gamma)$  across  $\gamma_0$ . Using the conservation laws for the linear and angular momenta one obtains the transformation between the precollisional and postcollisional velocities and spins:

$$\begin{aligned} \mathbf{v}_i &= \mathbf{v}'_i - \sigma_i \frac{1+e}{2} (\mathbf{k} \cdot \mathbf{g}'_{12}) \mathbf{k} + \sigma_i \frac{\kappa}{2} \frac{1+\beta(\gamma)}{1+\kappa} \mathbf{k} \times (\mathbf{k} \times \mathbf{g}'_{12}) \\ \mathbf{s}_i &= \mathbf{s}'_i + \frac{1}{2} \frac{1+\beta(\gamma)}{1+\kappa} \mathbf{k} \times \mathbf{g}'_{12} \end{aligned} \quad (9)$$

where  $i = 1, 2$ ,  $\sigma_1 = 1$  and  $\sigma_2 = -1$ . The Jacobian of this transformation is given by

$$J(\gamma) \equiv \left| \frac{\partial(\mathbf{v}_1, \mathbf{v}_2, \mathbf{s}_1, \mathbf{s}_2)}{\partial(\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{s}'_1, \mathbf{s}'_2)} \right| = \begin{cases} e\beta_0^2 & \gamma < \gamma_0 \\ e|\beta(\gamma)| & \gamma > \gamma_0 \end{cases}. \quad (10)$$

Let  $f(\mathbf{v}_1, \mathbf{s}_1, \mathbf{r}, t) (\equiv f_1)$  denote the single-particle distribution (of the velocity and spin) function at point  $\mathbf{r}$  and time  $t$ . The Boltzmann equation satisfied by  $f_1$  is

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \mathbf{v}_1 \cdot \nabla f_1 &= B(f, f; e, \beta_0, \mu) \\ &\equiv d^2 \int_{\mathbf{k} \cdot \mathbf{v}_{12} > 0} d\mathbf{v}_2 d\mathbf{s}_2 d\mathbf{k} (\mathbf{k} \cdot \mathbf{v}_{12}) \left( \frac{1}{eJ(\gamma)} f'_1 f'_2 - f_1 f_2 \right). \end{aligned} \quad (11)$$

As explained in the above, the choice of the hydrodynamic fields is dictated by the limit that defines the zeroth order solution. In the smooth elastic limit the conserved entities are the mass (or number), momentum and energy densities, and all the particles’ angular velocities (or spins). Therefore the hydrodynamic fields are the mass (or number) density, the momentum density (or velocity), the granular temperature (which is trivially related to the energy density), and all the spin-dependent number densities (i.e., the number density for a given value of the spin),  $n(\mathbf{s}, \mathbf{r}, t)$ . In other words, the system is considered to be a mixture, whose components are indexed by the respective values of their spins. To zeroth order (i.e., in the smooth limit) spins

can only diffuse around, hence  $n(\mathbf{s}, \mathbf{r}, t)$  obey diffusion equations. All in all, hydrodynamic fields are given by  $\mathbf{V}(\mathbf{r}, t) \equiv \frac{1}{n} \int \mathbf{v} f(\mathbf{v}, \mathbf{s}, \mathbf{r}, t) d\mathbf{v} d\mathbf{s}$ ,  $T(\mathbf{r}, t) \equiv \frac{1}{n} \int u^2 f(\mathbf{v}, \mathbf{s}, \mathbf{r}, t) d\mathbf{v} d\mathbf{s}$ , and  $\rho(\mathbf{s}, \mathbf{r}, t) \equiv \frac{1}{n} \int f(\mathbf{v}, \mathbf{s}, \mathbf{r}, t) d\mathbf{v}$ , where  $\mathbf{u}$  is the peculiar velocity,  $\mathbf{u} \equiv \mathbf{v} - \mathbf{V}(\mathbf{r}, t)$ , and  $n$  denotes  $n(\mathbf{r}, t)$ . Another field of interest is the velocity field corresponding to particles of spin  $\mathbf{s}$ . It is defined by  $\mathbf{V}(\mathbf{s}, \mathbf{r}, t) = \frac{1}{n\rho(\mathbf{s}, \mathbf{r}, t)} \int \mathbf{v} f(\mathbf{v}, \mathbf{s}, \mathbf{r}, t) d\mathbf{v}$ . The latter is not a hydrodynamic field, as it does not correspond to a conserved entity in the smooth elastic limit, and indeed it is expressible in terms of the hydrodynamic fields (see below), as an additional constitutive relation. A similar procedure to that used in the frictionless case now produces the continuum equations of motion from the (moments of the) Boltzmann equation. The form of the (continuum mechanics) equations for the velocity and temperature fields is the same as for smooth particles (the constitutive relations are different). In addition, the new field  $\rho$  satisfies

$$n \frac{D\rho(\mathbf{s})}{Dt} = \int B(f, f, \mathbf{v}, \mathbf{s}) d\mathbf{v} - \nabla \cdot (n\rho(\mathbf{s})\delta\mathbf{V}(\mathbf{s})) \quad (12)$$

where  $\frac{D}{Dt}$  is the material derivative. Also,  $\delta\mathbf{V}(\mathbf{s}, \mathbf{r}, t) = \mathbf{V}(\mathbf{s}, \mathbf{r}, t) - \mathbf{V}(\mathbf{r}, t)$  is the relative velocity between the particles of spin  $\mathbf{s}$  and the velocity field. The stress tensor is given by  $P_{\alpha\beta} = \int u_\alpha u_\beta f d\mathbf{u} d\mathbf{s}$ , the heat flux vector is  $Q_\alpha = \frac{1}{2} \int u_\alpha u^2 f d\mathbf{u} d\mathbf{s}$ , and the energy sink term is  $\Gamma = - \int d\mathbf{v}_1 d\mathbf{s}_1 \frac{v_1^2}{2} B(f, f, \mathbf{v}_1, \mathbf{s}_1)$ .

The small parameters used in the CE expansion for this problem are the Knudsen number,  $K$ , the degree of inelasticity,  $\epsilon$ , and  $\epsilon_3 \equiv 1 - \beta_0^2$ . Mathematically, in this limit one can choose the friction coefficient to be  $\mathcal{O}(1)$ , but this is not compatible with the smooth limit, hence we choose the value of  $\mu$  to be an additional small parameter. As this value affects the results only to third order in the small parameters, one may ignore the effect of  $\mu$  altogether (to second order) and therefore  $\beta$  can be taken to equal the constant  $\beta_0$  (which is assumed to be close to  $-1$ ).

The zeroth order distribution function,  $f_0$  (which solves the Boltzmann equation for  $\epsilon = \epsilon_3 = K = 0$ ), is given by  $f_0(\mathbf{u}, \mathbf{s}) = f_M(\mathbf{u})\rho(\mathbf{s})$ , where  $f_M(\mathbf{u}) = n(\frac{3}{2\pi T})^{\frac{3}{2}} e^{-\frac{3u^2}{2T}}$ . The full single-particle distribution function,  $f(\mathbf{v}, \mathbf{s})$ , can be written as  $f(\mathbf{v}, \mathbf{s}) = f_0(\mathbf{u}, \mathbf{s})(1 + \phi(\mathbf{u}, \mathbf{s}))$ . As in the frictionless case, it is assumed that  $\phi$  can be expanded in the small parameters:  $\phi = \phi_K + \phi_\epsilon + \phi_{\epsilon_3} + \phi_{K\epsilon} + \dots$ .

The perturbative solution of the Boltzmann equation, in powers of  $K$ ,  $\epsilon$ , and  $\epsilon_3$ , involves the repeated solution of  $\tilde{L}\phi_{\text{given order}} = R_{\text{same order}}$ , where  $R_{\text{same order}}$  depends on previous orders, and the linearized Boltzmann operator,  $\tilde{L}$ , is given by  $\tilde{L}\phi \equiv d^2 \int_{\mathbf{k} \cdot \mathbf{v}_{12} > 0} d\mathbf{v}_2 d\mathbf{s}_2 d\mathbf{k} (\mathbf{k} \cdot \mathbf{v}_{12}) f_0(\phi'_1 + \phi'_2 - \phi_1 - \phi_2)$ . The algebra involved in these inversions is rather heavy (the use of rather high orders in expansions in Sonine polynomials is required; convergence is obtained at sixth order in the polynomials). It has been carried out by employing the method described in appendix A. The application of the above CE expansion yields constitutive relations for the stress tensor, the heat flux, the energy sink, the field  $\delta\mathbf{V}$ , and the term  $\int B d\mathbf{v}$ . We have developed those constitutive relations to second (Burnett) order in the small parameters. As the results are rather cumbersome they are not presented here. Instead, we wish to mention our finding that the spin distribution functions for the HCS and the simple shear flow (the ones we calculated) are non-Maxwellian; both have power law tails. Interestingly, the distribution function for the simple shear flow is exactly expressible as  $\rho(\mathbf{s}) = A_0(B + C\zeta^2)^{-\frac{A}{2C}}$ , where  $s \equiv \|\mathbf{s}\|$ ,  $\zeta \equiv \frac{s}{\sqrt{T}}$  is the normalized spin, and  $A_0, B, C$  and  $A$  are numerical constants that depend on the collision parameters. Finally, as only the dilute case is considered here, the stress tensor (being the average of a symmetric entity) is symmetric, i.e., one does not obtain a micropolar theory [50].

### 3.3. Kinetic theory for steady states

There are several reasons for directly studying the steady state solutions of the Boltzmann equation, instead of, e.g., just considering the stationary solutions of the corresponding (time dependent) hydrodynamic equations. First, it is often simpler to consider ‘just’ stationary solutions. Furthermore, as it turns out, stationary solutions reveal much physics that the nearly automatic machinery of the CE method does not (at least not directly). In addition, it is convenient (and perhaps easier) to consider non-hydrodynamic limits (such as the Knudsen limit) employing the stationary solution. These points should become evident below.

*3.3.1. Simple shear flow in two dimensions.* Consider a collection of identical smooth discs, of diameter  $d$ , and unit mass, residing in a plane, whose (binary) collisions are characterized by a fixed coefficient of normal restitution,  $e$ . The collision rule is the same as in equation (1). The Boltzmann equation for a two-dimensional monodisperse collection of inelastically colliding discs, in a steady state, reads  $\mathbf{v}_1 \cdot \nabla f(\mathbf{v}_1) = \mathbf{B}(f, f, e)$ , where  $\mathbf{B}(f, f, e)$  is the nonlinear Boltzmann collision operator given by

$$\mathbf{B}(f, f, e) \equiv d \int_{\hat{\mathbf{k}} \cdot \mathbf{v}_{12} > 0} d\mathbf{v}_2 d\hat{\mathbf{k}} (\hat{\mathbf{k}} \cdot \mathbf{v}_{12}) \left( \frac{1}{e^2} f(\mathbf{v}'_1) f(\mathbf{v}'_2) - f(\mathbf{v}_1) f(\mathbf{v}_2) \right). \quad (13)$$

Let  $\beta \equiv \frac{1}{2T}$ . It is convenient to employ the following non-dimensionalization: the dimensionless velocity and position vectors are given by  $\tilde{\mathbf{v}} \equiv \sqrt{\beta} \mathbf{v}$  and  $\tilde{\mathbf{r}} \equiv \frac{\mathbf{r}}{\ell}$ , respectively, where  $\ell \equiv \frac{1}{2nd}$  is the mean free path. Let  $f \equiv n\beta \tilde{f}$  define the dimensionless single particle distribution function  $\tilde{f}$ . The Maxwellian distribution,  $\tilde{f}_0$ , is given by  $\tilde{f}_0 = \frac{1}{\pi} e^{-\tilde{v}^2}$  (recall that this is a 2D case). In a state of uniform density,  $f$  is a function of the peculiar velocity,  $\mathbf{u} \equiv \mathbf{v} - \mathbf{V}$ , alone. The rescaled Boltzmann equation reads  $(\tilde{\mathbf{u}}_1 + \tilde{\mathbf{V}}) \cdot \tilde{\nabla} \tilde{f}(\tilde{\mathbf{u}}_1) = \tilde{\mathbf{B}}(\tilde{f}, \tilde{f}, e)$ , where  $\tilde{\nabla}$  is a gradient with respect to the dimensionless coordinates,  $\tilde{\mathbf{V}}(\tilde{\mathbf{r}})$  is the dimensionless macroscopic velocity, and

$$\tilde{\mathbf{B}}(\tilde{f}, \tilde{f}, e) = \frac{1}{2} \int_{\hat{\mathbf{k}} \cdot \tilde{\mathbf{u}}_{12} > 0} d\hat{\mathbf{k}} d\tilde{\mathbf{u}}_2 (\hat{\mathbf{k}} \cdot \tilde{\mathbf{u}}_{12}) \left( \frac{1}{e^2} \tilde{f}(\tilde{\mathbf{u}}'_1) \tilde{f}(\tilde{\mathbf{u}}'_2) - \tilde{f}(\tilde{\mathbf{u}}_1) \tilde{f}(\tilde{\mathbf{u}}_2) \right). \quad (14)$$

Consider the shear flow given by  $\tilde{\mathbf{V}} = \tilde{\gamma} \tilde{y} \hat{\mathbf{x}}$ , where  $\tilde{\gamma} \equiv \sqrt{\beta} \ell \gamma_s$ ,  $\gamma_s$  being the shear rate. In this case (and assuming  $n$  and  $T$  to be uniform),  $\tilde{f}$  is a function of  $\tilde{\mathbf{u}}$  alone. Since for this flow  $\tilde{\mathbf{V}} \cdot \tilde{\nabla} \tilde{f}$  vanishes, the pertinent Boltzmann equation assumes the following form (below the tilde signs are omitted for simplicity):  $-\gamma u_{1y} \frac{\partial f}{\partial u_{1x}} = \mathbf{B}(f, f, e)$ , where  $u_{1x}$  and  $u_{1y}$  are the Cartesian components of  $\mathbf{u}_1$ .

As before, we specialize to the case of a slightly inelastic granular gas, i.e.  $\epsilon \equiv 1 - e^2$  is taken to be a small parameter. Unlike in the general unsteady case, here one cannot perform a perturbative expansion in  $\epsilon$  and the Knudsen number, since these two are related in the steady state. Therefore, one has to first understand this relation, then use it in order to construct a consistent perturbation theory. In the case at hand, recall from the introduction that the granular temperature is given by  $T = C \frac{\gamma^2 \ell^2}{\epsilon}$ . Consider now the double limit,  $\gamma \rightarrow 0$  and  $\epsilon \rightarrow 0$ , with the ratio  $\frac{\gamma^2}{\epsilon}$  kept fixed. In this limit one obtains a state of equilibrium (since there is no shear and no inelasticity), with the rescaled distribution being  $f_0 = \frac{1}{\pi} e^{-u^2}$ , while the value of the temperature is kept fixed. As explained in the above, this limit is not singular. Therefore, by defining a scaled variable,  $\gamma^*$ , by  $\gamma \equiv \sqrt{\epsilon} \gamma^*$ , and assuming that  $\gamma^*$  is  $\mathcal{O}(1)$ , one can obtain a systematic perturbative expansion in this case. Since  $\gamma^*$  can still have some dependence on  $\epsilon$ , it is convenient to write the distribution function as  $f(\mathbf{u}) \equiv f_0(u)(1 + \Phi(\mathbf{u}))$ , and take  $\Phi$  and  $\gamma$  to be represented by power series in  $\sqrt{\epsilon}$ :  $\Phi(\mathbf{u}) = \sqrt{\epsilon} \Phi_1(\mathbf{u}) + \epsilon \Phi_2(\mathbf{u}) + \dots$ , and

$\gamma = \gamma_1\sqrt{\epsilon} + \gamma_2\epsilon + \gamma_3\epsilon^{\frac{3}{2}} + \mathcal{O}(\epsilon^2)$ . Notice that the above definitions of the dimensionless shear rate and temperature immediately imply the following (dimensional) relation:

$$T = \frac{\ell^2\gamma_s^2}{2(\gamma_1\sqrt{\epsilon} + \gamma_2\epsilon + \dots)^2}. \quad (15)$$

A perturbative calculation [15] of the solution of the Boltzmann equation, that is based on the above ansatz for the distribution function, and  $\gamma$ , yields  $\gamma_1 \approx 0.8771$  and  $\gamma_2 = 0$ . These results agree, as expected, with those obtained from the full hydrodynamic description. It follows from the obtained distribution function that the stress tensor components are given by (in dimensionless units):  $\tau_{xx} = \frac{1}{2} + \epsilon\tilde{\tau}_{xx} + \mathcal{O}(\epsilon^2)$ ,  $\tau_{xy} = \tau_{yx} = \sqrt{\epsilon}\tilde{\tau}_{xy} + \mathcal{O}(\epsilon^{\frac{3}{2}})$  and  $\tau_{yy} = \frac{1}{2} + \epsilon\tilde{\tau}_{yy} + \mathcal{O}(\epsilon^2)$ , where  $\tilde{\tau}_{xx} = 0.2612$ ,  $\tilde{\tau}_{yy} = -0.2612$ , and  $\tilde{\tau}_{xy} = -0.3572$ . Consequently, the ratio of  $\tau_{xx}$  and  $\tau_{yy}$  is a universal function of  $\epsilon$ , given, to second order in the above perturbation theory, by

$$\frac{\tau_{xx}}{\tau_{yy}} \approx \frac{1 + 0.5224\epsilon}{1 - 0.5224\epsilon}. \quad (16)$$

As in the above, the anisotropy of the normal stresses is a Burnett effect (also, an  $\mathcal{O}(\epsilon)$  effect), i.e., it is not captured at the Navier–Stokes level (linear order in the gradients). Notice that the general form of this expression (i.e., without the substitution of the relation between the shear rate and the granular temperature) is correct for molecular fluids as well. If one does not substitute the relation between the granular temperature and shear rate, leading to equation (16), one obtains  $0.679\frac{\gamma_s^2\ell^2}{T}$  instead of  $0.5224\epsilon$  in equation (16), in agreement with the Burnett order result for an elastically colliding gas. In this case (following the above explanation) the ratio of normal stresses is very close to unity.

**3.3.2. The heat conducting granular gas, with gravity.** Consider a three-dimensional granular gas, composed of monodisperse spheres of diameter  $d$ , and unit mass, whose collisions are characterized by a fixed coefficient of normal restitution,  $e$ . As in the above, the degree of inelasticity is defined by  $\epsilon \equiv 1 - e^2$ . Assume that the gas is homogeneous in the  $xy$  plane, the projection of the velocity in the  $x, y$  plane is isotropically distributed, and the gas is subject to gravity in the  $-z$  direction. This corresponds, e.g., to a gas excited by a vibrating floor, in the absence of convection. The Boltzmann equation in this case is  $v_z\frac{\partial f}{\partial z} - g\frac{\partial f}{\partial v_z} = B(f, f; e)$ , where  $B$  is given by equation (13). The hydrodynamic equations are obtained (as usual) by multiplying the Boltzmann equation by  $1, \mathbf{v}$ , and  $\mathbf{v}^2$ , and integrating over the velocity. Since, by assumption, the macroscopic velocity field vanishes, the multiplication by  $1$  yields an identity,  $0 = 0$ . The multiplication by  $v_x$  or  $v_y$  yields similar identities (using the assumed isotropy in the  $x, y$  plane). The multiplication by  $v_z$  yields  $\frac{d(nT_z)}{dz} + ng = 0$ , where  $T_z \equiv \frac{1}{n} \int v_z^2 f \, d\mathbf{v}$ . If the velocity distribution were isotropic, one would have obtained  $T_z = \frac{1}{3}T$ , and the last equation would have yielded the standard barometric formula. The multiplication of the above stationary Boltzmann equation by  $v^2$ , yields  $2\frac{\partial Q}{\partial z} = -n\Gamma$ , where  $\Gamma$  is the energy sink, and the ( $z$ -component of the) heat flux is given by  $Q = \frac{1}{2} \int v_z v^2 f \, d\mathbf{v}$ . Note that this problem has several length scales: the mean free path,  $\ell$ , the ‘gravitational length’,  $\frac{T}{g}$ , and the hydrodynamic resolution (used to define the Knudsen number,  $K$ ),  $L$ . One could therefore devise a multiple-scale perturbation theory of the pertinent Boltzmann equation. However, by noticing (as in the previous subsection) that these scales are not independent one can devise a direct perturbation theory, using a single small parameter. To this end, note that since  $Q$  must be proportional to the gradient of the temperature and/or density (to  $\mathcal{O}(K)$ ) it follows from the balance of the spatial decay of the heat flux and the energy sink (which is proportional to  $\epsilon$ ) that  $K^2$  scales as

$\epsilon$ ; in addition, since the derivative of the (logarithm of) the granular temperature is proportional to  $g$ , gravity should be scaled with  $K$ . Therefore one can define the following scaling, using the Knudsen number,  $K \equiv \frac{\ell}{L}$ :  $\gamma K \equiv \frac{3g\ell}{T}$  (note that  $\gamma$  here is different than in the previous subsection) and  $\epsilon \equiv \delta K^2$ . With these definitions one can construct a perturbative expansion in powers of  $K$  alone (in which  $\gamma$  is taken to be  $\mathcal{O}(1)$ , as is  $\delta$ ). Notice that the above assumption precludes the case of ‘strong’ gravity, i.e., when the gain or loss of speed during a mean free time can be comparable to the thermal speed. Some technical details now follow. It is convenient to define  $D \equiv \ell u_z \frac{d}{dz}$ , scale the velocity by the local thermal speed  $\mathbf{u} = \sqrt{\frac{3}{2T}} \mathbf{v}$ , and the distribution function by the local density:  $f = n(\frac{3}{2T})^{\frac{3}{2}} \tilde{f}$ . As before, let  $f = f_0(1+\phi)$ , with  $f_0 = n(\frac{3}{2\pi T})^{\frac{3}{2}} e^{-u^2}$ . The scaled Maxwell distribution function is then  $\tilde{f}_0 = \pi^{-\frac{3}{2}} e^{-u^2}$ . Upon expanding  $\phi$  in powers of  $K$ ,  $\phi = \phi_K + \phi_{KK} + \dots$ , and using the above scaling, one generates a perturbation expansion for the pertinent (and rescaled) Boltzmann equation. At order  $K$  one obtains from the momentum (density) equation  $D \log n + D \log T + \gamma K u_z = 0$  (where  $D$  is  $\mathcal{O}(K)$ ), and from the Boltzmann equation  $L\phi_K = D \log T (u^2 - \frac{5}{2}) = u_z (u^2 - \frac{5}{2}) K \frac{d \log T}{dz}$ . This equation is similar to an equation obtained in the general time dependent problem [16], and its solution can be expressed as  $\phi_K = K \Phi_T(u^2) u_z \frac{d \log T}{dz}$  where  $\Phi_T$  is a function of the scalar  $u$ , calculated in [16]. The calculation of higher order terms is rather cumbersome and will not be presented here. Once the distribution function is known, it is straightforward to use the resulting constitutive relations to obtain the equation satisfied, e.g., by the temperature field. To second order in  $K$  one obtains

$$\frac{d^2 \sqrt{T}}{d\xi^2} - \frac{n\ell g}{p(\xi)} \frac{2\kappa - \lambda}{2(\kappa - \lambda)} \frac{d\sqrt{T}}{d\xi} - \frac{\tilde{\delta}}{4(\kappa - \lambda)} \sqrt{T} = 0 \quad (17)$$

where  $\kappa$  and  $\lambda$  can be read off the constitutive relation for the heat flux in the time dependent case. The pressure is given by  $p(\xi) = p_0 - (n\ell g)\xi$ , and  $\xi$  is a length measured in accumulated mean free paths:  $\xi \equiv \int_0^z \frac{dz'}{\ell(z')}$ . The solution of this equation depends on the boundary conditions [30].

### 3.4. Comments on extensions to low and high densities

In the case discussed in the previous section, the upper (large  $z$ ) part of the granular gas can be too dilute for the hydrodynamic equations to be valid (high  $K$ ). This is expected when the system has no lid or when the lid is sufficiently high. In this case the boundary conditions on the hydrodynamic equations are determined by the properties of the transition region between the low  $K$  and high  $K$  (or Knudsen) domains. While an effective boundary condition has been proposed for equation (17) (see [30]), it is still interesting (and important) to study the non-hydrodynamic domain and its ‘coupling’ to the hydrodynamic region. This is possible since the Boltzmann operator can be linearized (to leading order) in the hydrodynamic region, and it is negligible (as would be its linearization) in the Knudsen domain. Furthermore, it is easy to check that if  $f$  is (near) Maxwellian at, say  $z = z_0$  in the Knudsen domain, it remains (near) Maxwellian for  $z > z_0$  (with decreasing density). Therefore the ‘local equilibrium’ assumption still holds (for small  $\epsilon$ ), even in the Knudsen domain (and uniformly in the system). One needs therefore to solve the integrodifferential equation  $v_z \frac{\partial f}{\partial z} - g \frac{\partial f}{\partial v_z} = B(f_0, f_0, e = 1) + f_0 L\phi$  in the entire system. The solution of this equation can be obtained by employing a similar method to that used for the derivation of boundary conditions [51], where, as in this case, the hydrodynamic region borders on a Knudsen domain (the Knudsen layer). Further extensions of this idea should be useful for the study of the inner structure of shocks. These extensions are beyond the scope of this paper.

The extension of kinetics to the description of gases of moderate densities has proven non-trivial, even in the realm of molecular gases. A common phenomenological method (which has also been applied to granular gases, see in [1]) is the use of the Enskog–Boltzmann equation, in which the product of the distribution functions on the rhs of the Boltzmann equation is multiplied by a volume fraction dependent term (for a more accurate description see [25]) that models a correction to the molecular chaos assumption. The Enskog–Boltzmann equation begets hydrodynamic equations of motion which are quite reliable in the case of hard spheres, and are (at least qualitatively) in good agreement with MD simulations. However, their validity for granular gases remains questionable. A method which is in principle superior to this approach is based on response theory (see in [1]), which is not limited, in principle, to low densities. However, as its practical application requires the use of a gradient expansion, it is still limited to near elastic gases.

#### 4. Conclusion

It is still unclear why hydrodynamics should ‘work’ for granular gases in the first place (in practice, down to  $e = 0.6$  or so), given the lack of scale separation. The extension to moderate densities is still very much an open problem. It is also not completely clear why models which ignore friction yield results which are in good agreement with simulations and experiments. Incidentally, a good theory for frictional granular gases is yet to be developed. The Knudsen regime, boundary conditions, shocks and other problems need to be studied in depth. Patterns that are observed in granular gases [1] are nicely described by ‘amplitude equations’ [52]; a relation of the latter to granular hydrodynamics needs to be established. A better understanding of the instabilities of granular gases needs to be achieved. All of this and more also applies to granular mixtures [53], which are even less well understood than monodisperse granular gases. While the above list of open problems is far from exhaustive, we would like to mention just one more important problem, which would connect the ideal models used in theory (mostly spheres) to reality, namely the problem of describing the kinetics and hydrodynamics of non-spherical grains.

#### Acknowledgments

One of us (IG) gratefully acknowledges grant no 689/04 from the Israel Science Foundation (ISF), and grant no 1999-417 from the US–Israel Binational Science Foundation (BSF), for partial support of this work.

#### Appendix A. A computationally aided method for solving $L\phi = R$

A recurrent problem in the Chapman–Enskog expansion is the solution of equations of the type  $L\phi = R$ , where  $L$  is the linearized Boltzmann operator (corresponding to elastic collisions) and  $R$  is an  $n$ -dimensional tensor in the peculiar velocity  $\mathbf{u}$  (when frictional interactions are accounted for,  $R$  may also depend on the angular velocity, but this case is not treated below). A standard method for inverting  $L$  involves the expansion of  $\phi$  and  $R$  in a complete set of orthogonal polynomials, the weight function being the equilibrium distribution. This transforms the above equation into a matrix inversion problem. The dimension of the matrix is determined by the maximal order of the employed polynomials. While matrix inversion is easily carried out using standard software, the multi-dimensional integrals arising in the process are not trivial to calculate. It turns out that it is relatively easy to compute the action

of  $L$  on the generating function for the desired polynomials, and then take derivatives of the result. The high order derivatives one needs to take comprise a formidable task, if one needs to do so by hand. However, symbolic manipulators take derivatives very rapidly (and make no mistakes!). Here we demonstrate the method we have developed on the case of hard spheres, and the polynomials are chosen (as is standard in this problem) to be the Sonine polynomials; see appendix B. A different method was proposed in [54].

Consider e.g., the equation corresponding to the linear order in  $K$ , equation (7). As mentioned in the text following equation (7), the isotropy of  $L$  implies that  $\phi_K$  has the same tensorial form as  $R$ . The form of the solution is given in equation (8). Below we denote  $\hat{\Phi}_T(u^2) \equiv \hat{\Phi}_c(u)(u^2 - \frac{5}{2})$ . The linearity of  $L$  implies that equation (7) can be separated into two problems:  $L[\Phi_T(u^2)u_i] = (u_1^2 - \frac{5}{2})u_{1i} \equiv R_T$ , and  $L[\Phi_v(u^2)\overline{u_i u_j}] = \overline{u_{1i} u_{1j}} \equiv R_v$ . The choice of the specific Sonine polynomials to expand in depends on the rank of the tensor that appears on the right-hand side of the equation; it is convenient to expand in terms of Sonine polynomials  $S_m^r(u^2)$  with  $m = \frac{1}{2}$  for a scalar rhs,  $m = \frac{3}{2}$  for a vector rhs,  $m = \frac{5}{2}$  for a second order tensor rhs and so on. For the present example, the following expansions are convenient:  $\Phi_T(u^2) = \sum_{r=0}^N \hat{\Phi}_{Tr} S_{\frac{3}{2}}^r(u^2)$  and  $R_T(u^2) = \sum_{r=0}^N a_{Tr} S_{\frac{3}{2}}^r(u^2)$ . The projection of the above two equations on the appropriate polynomials is carried out by operating on them with  $\int u_{1i} e^{-u_1^2} S_{\frac{3}{2}}^p(u_1^2) d\mathbf{u}_1$  and  $\int \overline{u_{1i} u_{1j}} e^{-u_1^2} S_{\frac{5}{2}}^p(u_1^2) d\mathbf{u}_1$ , respectively (summing over repeated indices). One thus obtains

$$\sum_{r=0}^{\infty} \hat{\Phi}_{Tr} \int u_{1i} e^{-u_1^2} S_{\frac{3}{2}}^p(u_1^2) L \left[ S_{\frac{3}{2}}^r(u^2) u_i \right] \Big|_{\mathbf{u}=\mathbf{u}_1} d\mathbf{u}_1 = \sum_{r=0}^{\infty} a_{Tr} \int u_1^2 e^{-u_1^2} S_{\frac{3}{2}}^p(u_1^2) S_{\frac{3}{2}}^r(u_1^2) d\mathbf{u}_1$$

and

$$\sum_{r=0}^{\infty} \hat{\Phi}_{vr} \int \overline{u_{1i} u_{1j}} e^{-u_1^2} S_{\frac{5}{2}}^p(u_1^2) L \left[ S_{\frac{5}{2}}^r(u^2) \overline{u_i u_j} \right] \Big|_{\mathbf{u}=\mathbf{u}_1} d\mathbf{u}_1 = \frac{2}{3} \sum_{r=0}^{\infty} a_{vr} \int u_1^4 e^{-u_1^2} S_{\frac{5}{2}}^p(u_1^2) S_{\frac{5}{2}}^r(u_1^2) d\mathbf{u}_1, \tag{A.1}$$

where use has been made of  $\overline{u_{1i} u_{1j}} \overline{u_{1i} u_{1j}} = \frac{2}{3} u_1^4$ . Next, rewrite equations (A.1) in the form

$$\sum_{r=0}^{\infty} M_{pr}^{(T)} \hat{\Phi}_{Tr} = 2\pi \frac{\Gamma(p + \frac{5}{2})}{p!} \delta_{p,1}$$

$$\sum_{r=0}^{\infty} M_{pr}^{(v)} \hat{\Phi}_{vr} = \frac{4\pi}{3} \frac{\Gamma(p + \frac{7}{2})}{p!} \delta_{p,0}$$

where

$$M_{pr}^{(T)} = \int u_{1i} e^{-u_1^2} S_{\frac{3}{2}}^p(u_1^2) L \left[ S_{\frac{3}{2}}^r(u^2) u_i \right] \Big|_{\mathbf{u}=\mathbf{u}_1} d\mathbf{u}_1$$

$$M_{pr}^{(v)} = \int \overline{u_{1i} u_{1j}} e^{-u_1^2} S_{\frac{5}{2}}^p(u_1^2) L \left[ S_{\frac{5}{2}}^r(u^2) \overline{u_i u_j} \right] \Big|_{\mathbf{u}=\mathbf{u}_1} d\mathbf{u}_1.$$

The matrix elements of  $M_{pr}^{(T)}$   $M_{pr}^{(v)}$  are quite hard to calculate directly. It is convenient to define the following generating functions:

$$M^{(T)}(s, t) \equiv \sum_{p,r=0}^{\infty} s^p t^r M_{pr}^{(T)} = \frac{1}{\pi^{\frac{5}{2}}} (1-s)^{-\frac{5}{2}} (1-t)^{-\frac{5}{2}} \int_{\mathbf{k} \cdot \mathbf{u}_{12} > 0} d\mathbf{u}_1 d\mathbf{u}_2 e^{-u_1^2 \frac{1}{1-s}} e^{-u_2^2} \times \left[ (\mathbf{u}_1 \cdot \mathbf{u}'_1) e^{-u_1^2 \frac{s}{1-s}} + (\mathbf{u}_1 \cdot \mathbf{u}'_2) e^{-u_2^2 \frac{s}{1-s}} - u_1^2 e^{-u_1^2 \frac{s}{1-s}} - (\mathbf{u}_1 \cdot \mathbf{u}_2) e^{-u_2^2 \frac{s}{1-s}} \right]$$

$$M^{(v)}(s, t) = \frac{1}{\pi^{\frac{3}{2}}} (1-s)^{-\frac{7}{2}} (1-t)^{-\frac{7}{2}} \int_{\mathbf{k} \cdot \mathbf{u}_{12} > 0} d\mathbf{u}_1 d\mathbf{u}_2 d\mathbf{k} (\mathbf{k} \cdot \mathbf{u}_{12}) e^{-u_1^2 \frac{1}{1-s}} e^{-u_2^2} \\ \times \left[ ((\mathbf{u}_1 \cdot \mathbf{u}'_1)^2 - \frac{1}{3} u_1^2 u_1'^2) e^{-u_1^2 \frac{t}{1-t}} + ((\mathbf{u}_1 \cdot \mathbf{u}'_2)^2 - \frac{1}{3} u_1^2 u_2'^2) e^{-u_2^2 \frac{t}{1-t}} - \frac{2}{3} u_1^4 e^{-u_1^2 \frac{t}{1-t}} \right. \\ \left. - ((\mathbf{u}_1 \cdot \mathbf{u}_2)^2 - \frac{1}{3} u_1^2 u_2^2) e^{-u_2^2 \frac{t}{1-t}} \right].$$

The calculation of the last two integrals is further facilitated by defining a ‘super-generating’ function,  $J_0$ , that can be analytically calculated, and whose derivatives yield these integrals (and in turn appropriate derivatives of the integrals yield the desired matrix elements). To this end define the following generalized Gaussian integral:

$$J_0(a, b, c, d, x, y, z, \gamma, \beta) = \int_{\mathbf{k} \cdot \mathbf{u}_{12} > 0} d\mathbf{u}_1 d\mathbf{u}_2 d\mathbf{k} (\mathbf{k} \cdot \mathbf{u}_{12}) e^{-F}$$

where

$$F = au_1^2 + bu_2^2 + cu_1'^2 + du_2'^2 + \frac{x}{2}(\mathbf{u}_1 + \mathbf{u}'_1)^2 + \frac{y}{2}(\mathbf{u}_1 + \mathbf{u}'_2)^2 + \frac{z}{2}(\mathbf{u}_1 + \mathbf{u}_2)^2 \\ + \gamma u_{12}^2 + \beta (\mathbf{k} \cdot \mathbf{u}_{12})^2 = \left(a + \frac{x}{2} + \frac{y}{2} + \frac{z}{2}\right) u_1^2 + \left(b + \frac{z}{2}\right) u_2^2 + \left(c + \frac{x}{2}\right) u_1'^2 \\ + \left(d + \frac{y}{2}\right) u_2'^2 + x(\mathbf{u}_1 \cdot \mathbf{u}'_1) + y(\mathbf{u}_1 \cdot \mathbf{u}'_2) + z(\mathbf{u}_1 \cdot \mathbf{u}_2) + \gamma u_{12}^2 + \beta (\mathbf{k} \cdot \mathbf{u}_{12})^2.$$

The function  $F$  is positive definite, hence the above integral converges.

To calculate  $J_0(a, b, c, d, x, y, z, \gamma, \beta)$  it is convenient to transform to centre of mass and relative velocities  $\mathbf{u}_{\text{cm}} = \frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2)$  and  $\mathbf{u}_{12} = \mathbf{u}_1 - \mathbf{u}_2$  and define  $\mathbf{U} = \mathbf{u}_{\text{cm}} + g\mathbf{u}_{12} + h\mathbf{u}'_{12}$  where  $g \equiv \frac{a-b+y+x}{2\lambda}$ ,  $h \equiv \frac{c-d+x-y}{2\lambda}$ , and  $\lambda \equiv a + b + c + d + 2x + 2y + 2z$ . Since  $\mathbf{u}_{1,2} = \mathbf{U} - (g \mp \frac{1}{2})\mathbf{u}_{12} - h\mathbf{u}'_{12}$ , and  $\mathbf{u}'_{1,2} = \mathbf{U} - g\mathbf{u}_{12} - (h \mp \frac{1}{2})\mathbf{u}'_{12}$ , it follows that  $F = \lambda U^2 + \mu u_{12}^2 + \nu (\mathbf{k} \cdot \mathbf{u}_{12})^2$  where

$$\mu = \frac{1}{4} \left( a + b + c + d + 2x + 4y - \frac{1}{\lambda} (a - b + c - d + 2x) \right) \\ \nu = \frac{1}{2} \left( y - x + 2\beta + \frac{2}{\lambda} (a - b + y + x)(c - d + x - y) \right).$$

Using  $d\mathbf{u}_1 d\mathbf{u}_2 = d\mathbf{U} d\mathbf{u}_{12}$ , the integration over  $\mathbf{U}$ ,  $\mathbf{u}_{12}$  and  $\mathbf{k}$  is trivial. The result is

$$J_0(a, b, c, d, x, y, z, \gamma, \beta) = \frac{2\pi^{\frac{7}{2}}}{\lambda^{\frac{3}{2}} \mu (\mu + \nu)}.$$

Next, it is convenient to define the operators:  $\delta_a \equiv -\frac{\partial}{\partial a}$ ,  $\delta_b \equiv -\frac{\partial}{\partial b}$ ,  $\delta_c \equiv -\frac{\partial}{\partial c}$ ,  $\delta_d \equiv -\frac{\partial}{\partial d}$ ,  $\delta_x \equiv -\frac{\partial}{\partial x} - \frac{1}{2}(\delta_a + \delta_c)$ ,  $\delta_y \equiv -\frac{\partial}{\partial y} - \frac{1}{2}(\delta_a + \delta_d)$ , and  $\delta_z \equiv -\frac{\partial}{\partial z} - \frac{1}{2}(\delta_a + \delta_b)$ .

Using this notation, one can rewrite e.g.,  $M^v$  as follows:

$$M^{(v)}(s, t) = \frac{1}{\pi^{\frac{3}{2}}} (1-s)^{-\frac{7}{2}} (1-t)^{-\frac{7}{2}} \left[ \left( \delta_x^2 - \frac{1}{3} \delta_a \delta_c \right) J_0 \left( \frac{1}{1-s}, 1, \frac{t}{1-t} \right) \right. \\ + \left( \delta_y^2 - \frac{1}{3} \delta_a \delta_d \right) J_0 \left( \frac{1}{1-s}, 1, 0, \frac{t}{1-t} \right) - \frac{2}{3} \delta_a^2 J_0 \left( \frac{1}{1-s} + \frac{t}{1-t}, 1 \right) \\ \left. - \left( \delta_z^2 - \frac{1}{3} \delta_a \delta_b \right) J_0 \left( \frac{1}{1-s}, \frac{1}{1-t} \right) \right].$$

In the above, a set of zeros in the arguments of  $J_0$  is omitted for simplicity (e.g.,  $J_0(\frac{1}{1-s}, 1, \frac{t}{1-t})$  really means  $J_0(\frac{1}{1-s}, 1, \frac{t}{1-t}, 0, 0, 0, 0, 0, 0)$ . In addition, the derivatives are to be followed by a substitution of the actual values of the parameters  $a, b, c$ , and  $d$ , and zero for the rest of the parameters. In the present case:  $a = \frac{1}{1-s}$ ,  $b = 1$ ,  $c = \frac{t}{1-t}$ , and  $d = x = y = z = \gamma = \beta = 0$ .

As mentioned, the derivatives can easily be carried out by a symbolic processor. The order (in the Sonine expansion) to which the calculations can be carried out depends on the available memory. Our old PC could easily compute the eighth order for all terms we needed in the CE expansion.

The construction of the matrix from the generating function is not always as simple as in the above example. Consider, for instance, the case  $R(\mathbf{u}) = h(u^2)u_i u_j = \sum_{r=0}^{\infty} \hat{h}_r S_{\frac{1}{2}}^r(u_i^2)u_i u_j$ . The solution is not, as before, a scalar function multiplied by  $u_i u_j$  but a more general form:  $\phi(\mathbf{u}) = \phi^{(1)}(u)\delta_{ij} + \phi^{(2)}(u)\overline{u_i u_j}$ . However, a rather straightforward generalization of the above method can be applied in such cases as well.

## Appendix B. Sonine polynomials: a reminder

The Sonine polynomial  $S_m^n(x)$  is the coefficient of  $s^n$  in the expansion

$$(1-s)^{-m-1} e^{-x \frac{s}{1-s}} = \sum_{n=0}^{\infty} s^n S_m^n(x)$$

and belongs to a complete orthogonal set with the weight function  $e^{-x} x^m$ . The orthogonality relations read

$$\int_0^{\infty} e^{-x} x^m S_m^p(x) S_m^q(x) dx = \delta_{p,q} \frac{\Gamma(m+1+p)}{p!}.$$

## References

- [1] Goldhirsch I 2003 *Annu. Rev. Fluid Mech.* **35** 267–93
- [2] Luding S and Goldshtein A 2003 *Granular Matter* **5** 159–63
- [3] Campbell C S 1990 *Annu. Rev. Fluid Mech.* **22** 57–92
- [4] Goldhirsch I 1991 *Proc. DOE/NSF Mtg on 'The Flow of Particulates and Fluids (WPI, Worcester, MA)* pp 211–35
- [5] Goldhirsch I and Zanetti G 1993 *Phys. Rev. Lett.* **70** 1619–22
- [6] Goldhirsch I, Tan M-L and Zanetti G 1993 *J. Sci. Comput.* **8** 1–40
- [7] Tan M-L and Goldhirsch I 1997 *Phys. Fluids* **9** 856–69
- [8] Livne E, Meerson B and Sasorov P V 2002 *Phys. Rev. E* **66** 050301
- [9] Brey J J, Ruiz-Montero M J and Cubero D 1999 *Phys. Rev. E* **60** 3150–7
- [10] McNamara S and Young W R 1991 *Phys. Fluids A* **4** 496–504
- [11] McNamara S and Young W R 1994 *Phys. Rev. E* **50** 28–31
- [12] Kadanoff L P 1999 *Rev. Mod. Phys.* **71** 435–44
- [13] Tan M-L and Goldhirsch I 1998 *Phys. Rev. Lett.* **81** 3022–5
- [14] Goldhirsch I and Sela N 1996 *Phys. Rev. E* **54** 4458–61
- [15] Sela N, Goldhirsch I and Noskowitz S H 1996 *Phys. Fluids* **8** 2337–53
- [16] Sela N and Goldhirsch I 1998 *J. Fluid Mech.* **361** 41–74
- [17] Jin S and Slemrod M 2001 *Physica D* **150** 207–18
- [18] Standish R K 1999 *Phys. Rev. E* **60** 5175–8
- [19] Ernst M H and Dorfman J R 1975 *J. Stat. Phys.* **12** 311–61
- [20] Glasser B J and Goldhirsch I 2001 *Phys. Fluids* **13** 407–20
- [21] Rericha E C, Bizon C, Shattuck M D and Swinney H L 2002 *Phys. Rev. Lett.* **88** 014302
- [22] Goldshtein A and Shapiro M 1995 *J. Fluid Mech.* **282** 75–114
- [23] Soto R and Mareschal M 2001 *Phys. Rev. E* **63** 041303
- [24] Kogan M K 1969 *Rarefied Gas Dynamics* (New York: Plenum)
- [25] Chapman S and Cowling T G 1970 *The Mathematical Theory of Nonuniform Gases* (Cambridge: Cambridge University Press)
- [26] Cercignani C 1975 *Theory and Application of the Boltzmann Equation* (Edinburgh: Scottish Academic Press)
- [27] Brey J J, Dufty J W, Kim C S and Santos A 1998 *Phys. Rev. E* **58** 4638–53
- [28] Goldhirsch I, Noskowitz S H and Bar-Lev O 2003 *Granular Gas Dynamics* ed T Pöschel and N Brilliantov (Berlin: Springer) pp 37–63

- [29] Soto R, Mareschal M and Risso D 1999 *Phys. Rev. Lett.* **83** 5003–6
- [30] Brey J J, Ruiz-Montero M J and Moreno F 2001 *Phys. Rev. E* **63** 061305
- [31] Walton O R and Braun R L 1986 *Acta. Mech.* **63** 73–86
- [32] Lun C K K 1991 *J. Fluid Mech.* **223** 539–59
- [33] Johnson P C and Jackson R 1987 *J. Fluid Mech.* **176** 67–93
- [34] Abu-Zaid S and Ahmadi G 1993 *Powder Technol.* **77** 7–17
- [35] Cafiero R and Luding S 2000 *Physica A* **280** 142–7
- [36] Moon S J, Swift J B and Swinney H L 2004 *Phys. Rev. E* **69** 031301
- [37] Nasuno S, Kudrolli A and Gollub J P 1997 *Phys. Rev. Lett.* **79** 949–52
- [38] Luding S, Huthmann M, McNamara S and Zippelius A 1998 *Phys. Rev. E* **58** 3416–25
- [39] Huthmann M and Zippelius A 1998 *Phys. Rev. E* **56** R6275–8
- [40] Herbst O, Huthmann M and Zippelius A 2000 *Granular Matter* **2** 211–9
- [41] Luding S 1995 *Phys. Rev. E* **52** 4442–57
- [42] McNamara S and Luding S 1998 *Phys. Rev. E* **58** 2247–50
- [43] Jenkins J T and Richman M W 1985 *Phys. Fluids* **28** 3485–94
- [44] Lun C K K and Savage S B 1987 *J. Appl. Mech.* **54** 47–53
- [45] Jenkins J T and Zhang C 2002 *Phys. Fluids* **14** 1228–35
- [46] Dahler J S and Theodosopulu M 1975 *Adv. Chem. Phys.* **31** 155–229
- [47] Shukhman I G 1984 *Astron. J.* **61** 985–1004 (in Russian)
- [48] Araki S 1998 *Icarus* **76** 182–98
- [49] Reif F 1965 *Fundamentals of Statistical and Thermal Physics* (New York: McGraw-Hill)
- [50] Hayakawa H, Mitarai N and Nakanishi H 2002 *Phys. Rev. Lett.* **88** 174301
- [51] Goldhirsch I 1999 *Chaos* **9** 659–72
- [52] Blair D L, Aranson I S, Crabtree G W, Vinokour V, Tsimring L S and Joserand C 2000 *Phys. Rev. E* **61** 5600–10
- [53] Shinbrot T and Muzzio F J 2000 *Phys. Today* **53** (3) 25–30
- [54] Pöschel T and Brilliantov N 2003 *Granular Gas Dynamics* ed T Pöschel and N Brilliantov (Berlin: Springer) pp 131–62